Twisted topological structures related to M-branes II: Twisted Wu and Wu^c structures

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Abstract

Studying the topological aspects of M-branes in M-theory leads to various structures related to Wu classes. First we interpret Wu classes themselves as twisted classes and then define twisted notions of Wu structures. These generalize many known structures, including Pin$^-$ structures, twisted Spin structures in the sense of Distler-Freed-Moore, Wu-twisted differential cocycles appearing in the work of Belov-Moore, as well as ones introduced by the author, such as twisted Membrane and twisted String^c structures. In addition, we introduce Wu^c structures, which generalize Pin^c structures, as well as their twisted versions. We show how these structures generalize and encode the usual structures defined via Stiefel-Whitney classes.

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Contents

1 Introduction

M-theory has proven over the years to be a very rich sources of various mathematical structures. In previous work [18] [20] [21] we uncovered some geometric and topological structures related to M-branes in M-theory. This letter is a continuation of the work [21], where the emphasis is on twisted topological structures.

The main focus will be on Wu classes v_i, which are mod 2 characteristic classes that can be written as polynomials over $\mathbb{Z}_2$ in the Steifel-Whitney classes w_j of same and/or lower degrees. In the lowest two degrees these structures are familiar: v_1 is the obstruction to manifold orientation, and v_2 is the obstruction to having a Pin$^-$ structure. The latter structure has many interesting applications to low-dimensional topology [13]. In high dimensions, Wu classes have interesting applications to surgery problems. In degrees four and six, the Wu classes have applications to the M5-brane partition function [32] [11] and to topological aspects of type IIB string theory [32] [1], respectively.

What we do in this paper can be summarized as follows

1. We interpret Wu structures themselves already as twisted structures defined by the Stiefel-Whitney classes. These structures include Spin structures and Membrane structures [21]. This is done in section 3.

2. In section 4 we introduce the notion of *twisted Wu structure*, which in cohomological degree two is related to twisted Spin structure (cf. [31] [7]). In degree four this will be a generalization of twisted Membrane structure, introduced in [21].

3. We also introduce structures that we call *Wu^c structures* in section 5. In the appropriate degrees, these are generalizations of the Spin^c structure and of the String$^{K(\mathbb{Z},3)}$ structure (cf. [21]).

4. Finally, in section 6, we describe a twist for the Wu^c structures leading to *twisted Wu^c structures*. These are generalizations of twisted Spin^c (and twisted Pin^c) structures and twisted String$^{K(\mathbb{Z},3)}$ structures [21] in degree 3 and degree 7, respectively.

Throughout, we emphasize the motivation and the relation to M-branes via examples. These example also include (spacetime) M-theory and type II string theory.

2 Wu classes and Wu structures

We start by providing a basic description and properties of the Wu classes and Wu structures that will be used in later sections.

2.1 Wu classes

The Stiefel-Whitney classes in terms of the Wu classes. Let BO denote the classifying space of the stable orthogonal group $\mathrm{O} = \bigcup_{k=0}^{\infty} \mathrm{O}(k)$. The i-dimensional *universal Wu class* v_i is the element of $H^i(BO;\mathbb{Z}_2)$ defined inductively via the Steenrod square Sq^j as (cf. [16][25] [26])

$$v_0 = w_0 = 1 \quad \text{and} \quad w_i = v_i + Sq^1 v_{i-1} + \cdots + Sq^i v_0 \text{ if } i \geq 1. \tag{2.1}$$

These classes can also be defined for manifolds via the classifying map. Let M^n be a closed n-dimensional manifold and consider the action of the Steenrod square on the cohomology of the manifold $Sq^i : H^n(M;\mathbb{Z}_2) \to H^{n+i}(M;\mathbb{Z}_2)$. By Poincaré duality, there are unique classes $v_i \in H^i(M;\mathbb{Z}_2)$ satisfying

$$\langle v_i \cup x, [M] \rangle = \langle Sq^i x, [M] \rangle \tag{2.2}$$

for all $x \in H^{n-1}(M;\mathbb{Z}_2)$. Thus the relation is (cf. [17] [28])

$$v_0(M) = 1 \quad \text{and} \quad w_i(M) = v_i(M) + Sq^1 v_{i-1}(M) + \cdots + Sq^i v_0(M) \text{ if } i \geq 1. \tag{2.3}$$

That is, if f denotes the classifying map for the stable tangent bundle of M, then $f^* w_i = w_i(M)$ and $f^* v_i = v_i(M)$ if $i \geq 0$. Since $Sq^i x = 0$ for $i > n - i$, the classes vanish: $v_i = 0$ for $i > [\frac{n}{2}]$. The class

$$v = 1 + v_2 + \cdots + v_{[\frac{n}{2}]} \in H^*(M;\mathbb{Z}_2) \tag{2.4}$$

is the *total Wu class* of M. The total Stiefel-Whitney class $w = 1 + w_1 + w_2 + \cdots + w_n$ of M is determined by the *Wu formula* $w = Sq(v)$, where $Sq = 1 + Sq^1 + Sq^2 + \cdots$ is the total Steenrod square operation. The Wu formula can be expanded as $1 + w_1 + \cdots = Sq^0(1 + v_1 + \cdots) + Sq^1(1 + v_1 + \cdots) + \cdots$. Via $w_i = \sum_{j=0}^{i} Sq^{i-j} v_j$, the first few classes are

$$\begin{aligned}
w_1 &= Sq^0(1) + Sq^0(v_1) = v_1 \ , \\
w_2 &= Sq^2(1) + Sq^1(v_1) + Sq^0(v_2) = v_1^2 + v_2 \ , \\
w_3 &= Sq^3(1) + Sq^2(v_1) + Sq^1(v_2) + Sq^0(v_3) = Sq^1(v_2) + v_3 \ .
\end{aligned}$$

Since Stiefel-Whitney classes are more familiar than Wu classes, we are generally more interested in inverting the above relations.

The Wu classes in terms of the Stiefel-Whitney classes. The above relation, i.e. the Wu formula, for the Steifel-Whitney classes in terms of the Wu classes can be inverted to get the latter classes as polynomials in the former classes. We describe two ways of doing this. The first is to use the anti-automorphism (canonical conjugation) $\chi(\mathcal{A}(2))$ on the mod 2 Steenrod algebra $\mathcal{A}(2)$. The second way uses the Todd classes Td_i.

1. *Anti-automorphism:* One can invert the relation and write the total Wu class in terms of the total Stiefel-Whitney class

$$v(M) = \chi(Sq)w(M)\ . \tag{2.5}$$

The anti-automorphism χ is defined recursively using Thom's recursion formula $\sum_{i=0}^{n} Sq^i\chi(Sq^{n-i}) = 0$. Since the mod 2 Steenrod algebra is generated multiplicatively by the elements Sq^{2^n}, χ is determined completely by knowledge of $\chi(Sq^{2^n})$ for all n. The identity [29] $Sq^{2^n} + \chi(Sq^{2^n}) = Sq^{2^{n-1}}\chi(Sq^{2^{n-1}})$ and the formula $\chi(Sq^{2^n}) = Sq^{2^n} + \sum_{i=1}^{n-1}\prod_{i=1}^{n-1}\left(\prod_{j=1}^{i} Sq^{2^{n-j}}\right)Sq^{2^{n-i}}$ allow for explicit calculation of χ. There is, in fact, an explicit formula for individual Wu classes in terms of the Stiefel-Whitney classes [33]. It follows from the Wu formula that $v_n = \sum_{i=1}^{n}\theta^{n-i}w_i$, where $\theta^l = \chi(Sq^l) \in \mathcal{A}(2)$ is the conjugation of Sq^l in $\mathcal{A}(2)$ and is defined inductively by

$$\theta^l = Sq^l + \sum_{i=1}^{l-1} Sq^i\theta^{l-i} = Sq^l + \sum_{j=1}^{l-1}\theta^{l-j}Sq^j\ . \tag{2.6}$$

For $l = 0$, $\theta^0 = 1$ and for $l = 1$, $\theta^1 = Sq^1$. The odd-dimensional terms are given in terms of the even-dimensional ones via $\theta^{2n+1} = \theta^{2n}Sq^1$.

2. *Todd classes:* The second method is much easier and uses the relation between the Wu classes and the Todd classes, as polynomials in the Steiefel-Whitney classes (and *not* in the Chern classes), i.e. $\mathrm{Td}_n(w_1, \cdots, w_n)$. The formula is given by (see [10])

$$v_n \equiv 2^n \cdot \mathrm{Td}_n(w_1, \cdots, w_n) \mod 2\ , \tag{2.7}$$

through which the expansion of the Wu classes can be most efficiently read off from the corresponding expansion of the Todd genus. For example, for the low degree classes of most direct relevance to us, we have

$$\begin{aligned}
v_1 &= w_1\ ,\\
v_2 &= w_2 + w_1^2\ ,\\
v_3 &= w_1w_2\ ,\\
v_4 &= w_4 + w_3w_1 + w_2^2 + w_1^4\ ,\\
v_5 &= w_4w_1 + w_3w_1^2 + w_2^2w_1 + w_2w_1^3\ ,\\
v_6 &= w_4w_2 + w_4w_1^2 + w_3^2 + w_3w_2w_1 + w_3w_1^2 + w_2^2w_1^2\ .
\end{aligned} \tag{2.8}$$

Additivity property of Wu classes. From relation (2.5) , and using the formula $w(E\oplus F) = w(E)\cup w(F)$, we immediately see that $v(E\oplus F) = \chi(Sq)(w(E\oplus F)) = \chi(Sq)\,(w(E)\cup w(F))$, which is equal to $\chi(Sq)(w(E)) \cup \chi(Sq)(w(F))$ so that the Wu class is additive under Whitney sum, i.e., it satisfies

$$v(E\oplus F) = v(E)\cup v(F)\ . \tag{2.9}$$

Example 1. Consider the second Wu class. We have $v_2(E\oplus F)$ equals to $w_2(E\oplus F) + w_1(E\oplus F)\cup w_1(E\oplus F)$. Expanding, the first summand gives $w_2(E) + w_2(F) + w_1(E)w_1(F)$, while the second summand gives $w_1(E)^2 + w_1(F)^2 + 2w_1(E)w_1(F)$. Since w is a mod 2 class, we have $2w_1(E)w_1(F) = 0$. Therefore, altogether we have $w_2(E)+w_2(F)+w_1(E)^2+w_1(F)^2+w_1(E)w_1(F)$, which can be written as $v_2(E) + v_2(F) + v_1(E)v_1(F)$.

As we will consider structures defined by Wu classes, it would be useful to get some idea of when such classes vanish in relation to Stiefel-Whitney classes.

(Non)vanishing of the Wu classes. The Stiefel-Whitney numbers $w_{i_1} \cdots w_{i_r}[M^m]$, for $i_1 + \cdots i_r = m$, determine the cobordism class of the m-manifold M [27]. This implies, in particular, that if $n > 0$ and M is not a boundary then there must be an $i > 0$ for which $v_i \neq 0$. Let $B_k^* = H^*(BO; \mathbb{Z}_2)/I(v_i \mid i > k)$, where $I(v_i \mid i > k)$ is the ideal generated over the Steenrod algebra by the classes v_i, $i > k$. For $k = 1$, B_1^* has $v = 1 + v_1$ and $w = Sqv = 1 + v_1 + v_1^2$ with $0 = Sq^1(v_1^2) = Sq^1 w_2 = w_3 + w_2 w_1 = w_2 w_1 = v_1^3$. Thus $B_1^* = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with basis $1, v_1, v_1^2$. It can be shown by induction that B_k^* is finite-dimensional [28]. Contrary to what happens with Stiefel-Whitney classes, there is a closed manifold M^n which bounds with nonvanishing Wu class [28] $v = 1 + v_1 + v_2 + v_3$ for $n > 7$, $v = 1 + v_1 + v_2$ for $n > 5$, and $v = 1 + v_1$ for $n > 2$.

Example 2. Every closed seven-manifold M^7 which fibers over the sphere S^5 or S^6 is a boundary. That is there are bundles $\Sigma_2 \xrightarrow{i} M^7 \xrightarrow{\pi} S^5$ for which $v_i = 0$ for $i > 1$ and bundles $S^1 \xrightarrow{i} M^7 \xrightarrow{\pi} S^6$ for which $v_i = 0$ for $i > 0$. Similarly, every closed eleven-manifolds Y^{11} which fibers over the sphere S^j with $7 \leq j \leq 10$ is a boundary. In this case $v_i = 0$ for $i > 2$, $i > 1$, $i > 1$ and $i > 0$, corresponding respectively to the cases $j = 7, 8, 9$, and 10.

Odd-dimensional Wu classes. The odd-dimensional Wu classes can be written in terms of the lower Wu classes via the formula [33]

$$v_{2n+1} = \sum_{i \geq 1} (w_1)^{2^i - 1} v_{2n+2-2^i} \ . \tag{2.10}$$

This immediately implies that for oriented manifolds the odd-dimensional Wu classes are all zero. In fact, for $x \in H^{n-2i-1}(M)$, the same conclusion can be reached from the relation $Sq^{2i+1}(x) = Sq^1 Sq^{2i}(x) = v_1 \cup Sq^{2i}(x) = w_1 \cup Sq^{2i}(x)$.

Example 3. Special class of manifolds. Consider manifolds M whose total Stiefel-Whitney class $w(M)$ has nonzero components only in degrees that are powers of 2, i.e. it satisfies the condition $w(M) = 1 + \sum_{j \geq 1} w_{2^{j-1}}$. This is satisfied by a class of manifolds which include projective spaces. For example, for real projective spaces we have $w(\mathbb{R}P^5) = 1 + a^2 + a^4$, $w(\mathbb{R}P^9) = 1 + a^2 + a^8$ and $w(\mathbb{R}P^{11}) = 1 + a^4 + a^8$, where a is the class of the real classifying line bundle. For such manifolds, the Wu classes have the explicit form [33]

$$\begin{aligned}
v_i &= \sum_{j=1}^{m} (w_{2^{j-1}})^{2^{m-j}} && \text{if } i = 2^{m-1} \geq 1 \ , \\
v_i &= \sum_{j=1}^{m} \sum_{k=m+1}^{n} (w_{2^{j-1}})^{[(i-2^{k-1})/2^{j-1}]} (w_{2^k})^{2^{k-j-1}} && \text{if } i = 2^{m-1} + 2^{n-1} \text{ with } n > m \geq 1 \ , \\
v_i &= 0 && \text{otherwise.}
\end{aligned}$$

Indeed for the real projective space $\mathbb{R}P^n$, the total Wu class is $v(\mathbb{R}P^n) = \sum_{i=0}^{n} \binom{n-i}{i} a^i$ where a is the class of the classifying real line bundle. We will discuss the significance of powers of the Steifel-Whitney classes (especially squares) in our context in section 5.

2.2 Wu structures via classifying spaces

Consider the principal fibration $BO[v_{2k}]$ over the classifying space BO of the orthogonal group, with fiber the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2k-1)$, and with Postnikov invariant of the fibration equal to the Wu class $v_{2k} \in H^{2k}(BO; \mathbb{Z}_2)$. Given a fixed classifying map $f : X \to BO$ which represents a vector bundle ξ over X, by a Wu structure on ξ we mean a lifting $\tilde{f} : X \to BO[v_{2k}]$ of the map f, $f = p \circ \tilde{f}$, where p is the projection of the fibration. The obstruction is obviously $v_{2k}(\xi) = f^*(v_{2k})$. If it vanishes then the set of all Wu structures on ξ, denoted by $Wu(\xi)$ or $Wu(f)$ obtains the natural structure of an affine space over $H^{2k-1}(X; \mathbb{Z}_2)$.

Definition 1 ([3] [15]) *A Wu structure on a space M is a lifting of the classifying space map $f : M \to BO$ to the connected cover $BO[v_i]$ obtained from BO by killing the class v_i. We have the following diagram*

$$\begin{tikzcd} & BO[v_i] \arrow[d, "\pi"] \\ M \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"] & BO \end{tikzcd} \quad . \tag{2.11}$$

The space $BO[v_i]$ is a principal fibration over BO with fiber the Eilenberg-MacLane space $K(\mathbb{Z}_2, i-1)$

$$\begin{tikzcd} K(\mathbb{Z}_2, i-1) \arrow[r, "="] \arrow[d] & K(\mathbb{Z}_2, i-1) \arrow[d] \\ BO[v_i] \arrow[r] \arrow[d, "\pi"] & EK(\mathbb{Z}_2, i) \arrow[d] \\ BO \arrow[r, "k"] & K(\mathbb{Z}_2, i) \end{tikzcd} \quad , \tag{2.12}$$

where the k-invariant of the fibration is an element v_i in the cohomology $H^i(BO; \mathbb{Z}_2)$ defined by the ith Wu class v_i of the universal bundle over BO.

Wu structures can be induced from other structures; for instance

(1) *Spin structure*: A Spin structure leads to a Wu structure, because of the existence of the map $B\mathrm{Spin} \to BO[v_k]$. In fact, this holds for many structures, including a framing and all connected covers $\mathcal{F}$ of the orthogonal group such as String structure. This is captured by the commutative diagram

$$\begin{tikzcd} B\mathcal{F} \arrow[rr, "q"] \arrow[dr] & & BSO[v_k] \arrow[dl, "\pi"] \\ & BSO & \end{tikzcd} \quad . \tag{2.13}$$

(2) *Dimension*: Because of the fact that Wu classes vanish in degree above half the dimension of the manifold, we can have a Wu structure in the right degrees simply by this dimension argument. For example, on $(8k+2)$-dimensional spaces, the Wu class v_{4k+2} is always zero [3]. This is useful in studying the M5-brane as well as type IIB string theory (cf. [22]).

3 Wu structures as twisted structures

The idea of this section is that the Wu structures themselves already can be interpreted as twisted structures corresponding to the Stiefel-Whitney classes of the same degree. This provides connections to various other structures. We describe the main point as follows. Rewrite relation (2.1) as

$$v_i = w_i - (Sq^1 v_{i-1} + \cdots + Sq^i v_0) := w_j - \alpha_i \quad \text{for } i = 2^j \ , \tag{3.1}$$

so that the terms involving the lower degree Wu classes can be interpreted as a twist for the structure defined by $w_i = 0$. In low degrees, we see that this corresponds to twisted Spin structure (for $j = 1$) and twisted Membrane structure (for $j = 2$). With this point of view, our definition is

Definition 2 *Let (X, α_i) be a compact topological space with a degree i cocycle $\alpha_i : X \to K(\mathbb{Z}_2, i)$ with $i = 2^j$. A Wu structure over X is a quadruple (M, ν, ι, η), where*
(1) M is a smooth compact oriented manifold together with a fixed classifying map of its stable normal bundle $\nu : M \to BO$;
(2) $\iota : M \to X$ is a continuous map;
(3) η is an α-twisted structure on M defined by $w_i(M) + \iota^* \alpha_i = 0$, that is, a homotopy commutative diagram

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & BO \arrow[d, "w_i"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\alpha_i"'] & K(\mathbb{Z}_2, i) \end{tikzcd} \quad , \tag{3.2}$$

where η is a homotopy between $w_i \circ \nu$ and $\alpha \circ \iota$.

Remarks. **1.** The above definition is inspired by definitions of other related structures in [31] [24] [21].
2. In the above definition, we can replace $K(\mathbb{Z}_2, i)$ by the product $K(\mathbb{Z}_2, i_1) \times K(\mathbb{Z}_2, i_2) \times \cdots \times K(\mathbb{Z}_2, i_r)$, where $i_1 + \cdots + i_r = i$. This product maps to $K(\mathbb{Z}_2, i)$.
3. Two Wu structures in this sense will be equivalent on M if there is a homotopy between the corresponding homotopies η and η'.

We record the idea above in the following

Proposition 3 *A Wu structure in degree 2^i is a twisted structure for the structure defined by the Stiefel-Whitney class in the same degree.*

Our main example will be a Pin^- structure (see [13]), since a v_2-structure is a Pin^- structure. The second Wu class v_2 is equal to the combination of Stiefel-Whitney classes $w_2 + w_1^2$. If a given manifold M is not Spin then $w_2(M) \neq 0$. Since this takes values in $\mathbb{Z}_2$, the only nonzero value is 1. This means that one can add to $w_2(M)$ a $\mathbb{Z}_2$-class α_1, in this case equal to $w_1(M)^2$, such that $v_2 = w_2(M) + \alpha_1 = 0 \in H^2(M; \mathbb{Z}_2)$.

Pin$^-$ structures. A Pin$^-$ structure on M is equivalent to a Spin structure on $TM \oplus \det(TM)$. The obstruction for existence of a Pin$^-$ structure on M is the characteristic class $w_2(M)+w_1(M)^2$. If M admits a Pin$^-$ structure, then the set of such structures Pin$^-(TM)$ is acted upon freely and transitively by $H^1(M;\mathbb{Z}_2)$. In more detail, consider the two nontrivial central extensions $p_\pm : \mathrm{Pin}(n)^\pm \to O(n)$ of the orthogonal group $O(n)$ by $\mathbb{Z}_2$. A Pin$^\pm$ structure on a vector bundle E is a lifting of the structure group from $O(n)$ to its double cover Pin$(n)^\pm$. The short exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{Pin}^\pm(n) \xrightarrow{p^\pm} O(n) \to 1$ gives rise to an exact sequence

$$H^0(X, O(n)) \xrightarrow{\delta^0} H^1(X;\mathbb{Z}_2) \to H^1(X, \mathrm{Pin}^\pm(n)) \xrightarrow{(p^\pm)^*} H^1(X, O(n)) \xrightarrow{\delta} H^2(X;\mathbb{Z}_2)\ . \tag{3.3}$$

Applying the classifying functor B gives $B\mathrm{Pin}^\pm(n) \xrightarrow{Bp^\pm} BO(n) \xrightarrow{\omega} K(\mathbb{Z}_2, 2)$. Now a classifying map $f_E : X \to BO(n)$ of an $O(n)$ bundle E has a lift to $B\mathrm{Pin}^\pm(n)$ if and only if $\omega \circ f_E$ is homotopic to zero. Since $[X, K(\mathbb{Z}, 2)] \cong H^2(X;\mathbb{Z}_2)$, this is true when the the generators of $H^2(BO(n);\mathbb{Z}_2)$ pull back to zero in $H^2(X;\mathbb{Z}_2)$. These pullback classes are $w_2(E)$ and $w_2(E) + w_1(E)^2$. The first corresponds to Pin$^+$ structures and the second to Pin$^-$ structures. See [13] for more details.

Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be a short exact sequence of real vector bundles. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. If E_i is Spin and E_j is Pin$^\epsilon$, E_k has a natural Pin$^{-\epsilon}$ structure. On the other hand, if E_i is Pin$^\epsilon$, E_j is Pin$^{-\epsilon}$, and E_k is orientable, E_k has a natural Spin structure.

Example 4. Let L be the real classifying line bundle over the real projective space $\mathbb{R}P^m$ and let $x = w_1(L)$ be the generator of $H^1(\mathbb{R}P^m;\mathbb{Z}_2) = \mathbb{Z}_2$. Then $T(\mathbb{R}P^m) \oplus 1 = (m+1)\cdot L$, $w_1(\mathbb{R}P^m) = (m+1)\cdot x$ and $w_2(\mathbb{R}P^m) = \frac{1}{2}m(m+1)\cdot x^2$. Then $\mathbb{R}P^{4k+2}$ and $\mathbb{R}P^{4k+3}$ are both Pin$^-$. In fact, each admits two Pin$^-$ structures. However, $\mathbb{R}\mathrm{P}^{4k+3}$ is also Spin but $\mathbb{R}P^{4k+2}$ is not, since even-dimensional real projective spaces are not orientable.

We have seen in section 2.2 that Wu structures can result from Spin and other structures. Here we illustrate the point for the Pin$^-$ case. This will require replacing BSO by BO in diagram (2.13).

Wu structures from Pin$^-$ structures. Using the Adem relation $Sq^{4k+2} = Sq^2(Sq^{4k}+Sq^{4k-1}Sq^1)$, in $H^*(BO;\mathbb{Z}_2)$ the following equality holds [15]

$$v_{4k+2} = \sum_{i=1}^{r} a_i \cdot Sq^{I_i} v_2 \tag{3.4}$$

for certain classes $a_i \in H^*(BO;\mathbb{Z}_2)$ and certain Steenrod operations $Sq^{I_i} \in \mathcal{A}(2)$, the mod 2 Steenrod algebra, and $r \geq 1$. This gives a universal Wu structure for Pin$^-$ bundles, so there exist Wu structures on the universal vector bundle over $B\mathrm{Pin}^-$. As described in [9], one of them can be fixed by taking a map $q : B\mathrm{Pin}^- \to BO[v_{4k+2}]$ that completes the commutative diagram

$$\begin{tikzcd} B\mathrm{Pin}^- \arrow[rr, "q"] \arrow[dr] & & BO[v_{4k+2}] \arrow[dl, "\pi"] \\ & BO & \end{tikzcd} \tag{3.5}$$

with the natural projection $B\mathrm{Pin}^- \to BO$. Now, given a vector bundle ξ over X and its classifying map $f : X \to BO$, we have the affine map $q_f : \mathrm{Pin}^-(\xi) \to Wu(\xi)$ which sends each lifting

$\tilde{f} : X \to B\mathrm{Pin}^-$ to $q \circ \tilde{f}$. Here $\mathrm{Pin}^-(\xi)$ and $\mathrm{Wu}(\xi)$ are, respectively, the set of Pin^- structures and Wu structures on the bundle ξ.

Special cases. When the composite Stiefel-Whitney classes are zero, the Wu class coincides with the Stiefel-Whitney class in that degree – at least for degrees that are powers of 2. In low degrees, Wu structures reduce to

1. *Spin structure*: Here $v_2 = w_2$ when $w_1 = 0$.
2. *Membrane structure*: We have $v_4 = w_4$ when $w_1 = w_2 = 0$. This corresponds to the first Spin characteristic class $Q_1 = \frac{p_1}{2}$, which is half the first Pontrjagin class p_1, being divisible by 2 [21].
3. *Dual membrane structure*: In this case, $v_8 = w_8$, when $w_1 = w_2 = w_4 = 0$. This corresponds to the second Spin characteristic class $Q_2 = \frac{p_2}{2}$, which is half the second Pontrjagin class p_2, being divisible by 2

$$Q_2 = w_8 \mod 2\ . \tag{3.6}$$

Working 2-locally, the vanishing of this class also means we have a Fivebrane structure [23].

4 Twisted Wu structures

A twisted Wu structure will be essentially defined by the condition $v + \alpha = 0$. More specifically, a twisted $\mathrm{Wu}(i)$ structure is defined by the condition $v_i + \alpha_i = 0$, where $\alpha_i \in H^i\mathbb{Z}_2$. Thus, a twisted $\mathrm{Wu}(i)$ structure on a manifold X is described by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\alpha_i"'] & BO[v_j] \arrow[d, "v_i"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & K(\mathbb{Z}_2, i) \end{tikzcd}\ , \tag{4.1}$$

where η is a homotopy between the map representing the class v_i and the cocycle α_i. More generally, we can also define such a twisted notion of Wu structure on a brane M.

Definition 4 *Let (X, α_i) be a compact topological space with a degree i cocycle $\alpha : X \to K(\mathbb{Z}_2, i)$. A twisted Wu structure over X is a quadruple (M, ν, ι, η), where*
(1) M is a smooth compact oriented manifold together with a fixed classifying map of its stable normal bundle $\nu : M \to BO$;
(2) $\iota : M \to X$ is a continuous map;
(3) η is an α-twisted structure on M defined by $v_i(M) + \iota^*\alpha_i = 0$, that is, a homotopy commutative diagram with a map $\iota : M \to X$

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & BO[v_j] \arrow[d, "v_i"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\alpha_i"'] & K(\mathbb{Z}_2, i) \end{tikzcd}\ , \tag{4.2}$$

where η is a homotopy between $v_i \circ \nu$ and $\alpha \circ \iota$.

Remarks. **1.** We have used $BO[v_j]$ to indicate that we might have some structure arising from Wu-connected cover of BO and not just BO itself (although in applications, the latter is more dominant).

2. When all the composite Stiefel-Whitney classes are zero (or all Wu classes v_j for $j < i = 2^k$ are zero) this reduces to $w_i + \alpha_i = 0$. Then, for $i = 2$ and 8, we have a twisted Spin structure [31] and a twisted Membrane structure [21], given respectively by $w_2 + \alpha_2 = 0$ and $w_4 + \alpha_4 = 0$.

Example 5. Twisted Pin$^-$ structures. Let us consider the degree two case. We know that v_2 is the obstruction for a Pin$^-$ structure. When a manifold does not admit a Pin$^-$ structure, this means that $v_2 \neq 0$. Since the Wu class is valued in $\mathbb{Z}_2$, being nonzero means it is $1 \in \mathbb{Z}_2$. This implies that adding a nonzero class in $H^2(M;\mathbb{Z}_2)$ will lead to a zero class for the sum, because this will be 2-torsion. Therefore, if a manifold does not admit a Pin$^-$ structure then it certainly admits what we call a *twisted Pin$^-$ structure* (cf. definition 5 below). Examples of this include the real projective spaces $\mathbb{RP}^{4k}$ and $\mathbb{RP}^{4k+1}$.

Definition 5 *A twisted Pin$^-$ structure is a twisted Wu structure in the sense of definition 4 for $i = 2$.*

Example 6. Type IIA superstring theory with a B-field on an orientifold. An orientifold is roughly a smooth manifold together with an involution and is encoded in a double cover $\pi : X_\omega \to X$ of orbifolds, where $\omega \in H^1(X;\mathbb{Z}_2)$ is the equivalence class of the double cover. A B-field in this context is a differential cohomology class taking values in a certain Postnikov truncation of connected real K-theory *ko* [7]. For type IIA string theory, one of the main results of [7] is that the first and second Stiefel-Whitney classes of 10-dimensional orientifold spacetime X are $w_1(X) = \omega$ and $w_2(X) = \omega^2 + a(B) \cup \omega$, where $a(B) \in H^1(X;\mathbb{Z}_2)$ is a certain class which accounts for the presence of the B-field. Now we see that if $\omega \neq 0$ and $a(B) \neq 0$, i.e. if we have a nontrivial orientifold and B-field, then the two expressions can be combined to give $w_2(X) + w_1(X)^2 = a(B) \cup \omega$. We can write this as $v_2 + \alpha_2 = 0$, so that this structure indeed corresponds to a twisted Wu(2)-structure; in fact it corresponds to the special case of twisted Pin$^-$ structure described above.

Observation 1 *A twisted Spin structure in the sense of [7] is a twisted Wu structure in the sense of this paper, with the twist provided by both the orientifold double cover and the B-field.*

Example 7. Type IIB string theory and twisted differential cocycles. The self-dual field in type IIB string theory is delicate and requires some machinery to describe. In [1] this is given by a Chern-Simons functional. Such a description requires dealing with differential integral Wu classes $\check{\lambda}$ which are elements of the category $\check{\mathcal{H}}^6_v$ of ν-twisted differential 6-cocycles. This is a torsor for differential characters $\check{\mathcal{H}}^6$ and whose objects are differential cocycles such that $v_6 = a(\check{\lambda})$ mod 2, where $a(\check{\lambda})$ is the characteristic class of $\check{\lambda}$. Instead of viewing this as a character twisted by the Wu structure, we will turn it around and view it as a Wu structure twisted by the character. The reason for this is that the Wu class defines a bundle(-like) structure, which should be the 'main' structure, while the character is a slight modification. This is analogous to treating the $\frac{1}{4}p_1$-shifted differential characters of [6] as twisted String structure in [24].

Observation 2 *A v-twisted differential character in the sense of [1] is an instance of a twisted Wu structure in the sense of this paper.*

5 Wu^c structures

In this section we consider structures coming from Wu classes on which the action of the Bockstein vanishes. This Bockstein operation $\beta : H^i(X;\mathbb{Z}_2) \to H^{i+1}(X;\mathbb{Z})$ is associated with the exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. We define a Wu^c structure as an integral lift of a Wu structure. Therefore, a Wu^c class is defined via the Bockstein on the corresponding Wu class, that is

Definition 6 *(i). A Wu^c characteristic class is given by $V_{i+1} := \beta v_i \in H^{i+1}(BO;\mathbb{Z})$. This is the obstruction to having a Wu^c structure.*
(ii). A Wu$(i)^c$ structure on a manifold M is defined by $V_{i+1}(M) := \beta v_i(M) = 0 \in H^{i+1}(M;\mathbb{Z})$, with $v_i(M) = f^* v_i$ a pullback via the classifying map of the universal Wu class. That is, v_i is the modulo 2 reduction of an integral class.

Remarks. **1.** The Wu^c characteristic class coincides with the integral Stiefel-Whitney class when the Wu class is indecomposable.

2. As in the case for integral Stiefel-Whitney classes, the Wu^c classes are also mostly relevant in the odd-dimensional case, i.e. for j odd in V_j. However, for β_2 the Bockstein homomorphism coming from the exact coefficient sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ instead of the above sequence, the condition $w_1(M)^2 = 0$ is equivalent to $\beta_2(w_1(M) = 0$ and has application to manifolds of dimension $4k+1$ [8].

Special cases. We consider specializations of the Wu^c structure depending on the dimension. We will need to understand the action of the Bockstein β acting on products of Stiefel-Whitney classes. Let us start with the case when this product is a square. Note that β is an integral lift of Sq^1, which is a derivation and hence $Sq^1(x^2) = Sq^1(x)x + xSq^1(x) = 2xSq^1(x) = 0$. This means that $\beta(x^2)$ must be a class that maps to 0 when reduced modulo 2. From the sequence $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}_2 \xrightarrow{\beta} \mathbb{Z}$ we see that if $\beta(x^2) = 0$ mod 2, it must be twice a class, i.e. $\beta(x^2) = 2y$ for some integral class y. Furthermore, y cannot be $\beta(z)$ for any mod class z, except 0, since $2\beta(z) = 0$. Now since our classes are Stiefel-Whitney classes, we see that $\beta(w_2^2) = 0$ since $w_2^2 = \rho_2(p_1)$ and that $\beta(w_4^2) = 0$ since $w_4^2 = \rho_2(p_2)$. Therefore, squares of Stiefel-Whitney classes vanish in our range of dimensions in the oriented cas.

1. A Wu$(2)^c$ structure is defined when $V_3 = \beta v_2 = \beta(w_2 + w_1^2) = 0$, that is by $W_3 + \beta(w_1^2) = 0$. If we impose orientation then this immediately gives a Spin^c structure. However, in full generality, the Wu$(2)^c$ structure can be interpreted as a Pin^c structure (see below).

2. A Wu$(4)^c$ structure is defined, in the oriented case, when $V_5 = \beta v_4 = \beta(w_4 + w_2^2) = 0$. When we have the Spin condition in addition, this reduces to the condition $\beta w_4 = 0$, which is the obstruction to a Membrane^c structure, introduced in [21]. Without the Spin requirement, a Wu$(4)^c$ structure might seem be a a twisted Membrane^c structure (defined in [21]) with the twist given by the integral class $\beta(w_2^2)$. However, as we saw above, w_2^2 admits an integral lift, and hence the would be twist $\beta(w_2^2)$ is zero. We can consider this in more generality: the class $w_4 + w_2^2$ is in fact the mod 2 reduction of the class which defines a String^c structure [20] (see below). An application for complex manifolds [5] gives $c_2 - c_1^2$ mod $2 = w_4 + w_2^2 \in H^4(X;\mathbb{Z}_2)$.

3. A $\mathrm{Wu}(6)^c$ structure is defined, in the oriented case, by imposing $V_7 = \beta v_6 = \beta(w_2 w_4) = 0$. Obviously, when we have either a Spin structure or a Membrane structure, then this condition is satisfied. In general, the Wu formula gives $w_6 = Sq^2 w_4 + w_2 w_4$. At the level of Chern classes, this says that $\rho_2 c_3 = Sq^2(\rho_2 c_2) + \rho_2(c_1 c_2)$ mod 2.

We now summarize the main points above in the following

Proposition 7 (Properties of $\mathbf{Wu}^c$ structures) .
1. *A $Wu(4)^c$ structure is the same as a Membrane^c structure.*
2. **(i)** *A Spin structure implies a $Wu(6)^c$ structure.*
(ii) *A Membrane structure implies a $Wu(6)^c$ structure.*
3. *The obstructions to Wu^c structures are additive for bundles, i.e. $(\beta v)(E \oplus F) = \beta v(E) + \beta v(F)$.*

Application. $\mathbf{Pin}^c$ structure. Let $\rho_2 : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ be the mod 2 reduction. By definition, a bundle E is Pin^c if and only if $w_2(E) \in \mathrm{image}(\rho_2)$, that is $W_3(E) = \beta w_2(E) = 0$ [13]. As $w_1(E)^2 = \rho_2(c_1(E \otimes \mathbb{C}))$, $w_2(E) \in \mathrm{image}(\rho)$ if and only if $w_2(E) + w_1(E)^2 \in \mathrm{image}(\rho_2)$ Note that if we have a sequence of bundles $0 \to E_1 \to E_2 \to E_3 \to 0$ with E_i Spin^c and E_j Pin^c, for $\{i,j,k\}$ a permutation of $\{1,2,3\}$, then E_k has a natural Pin^c structure. On the other hand, if E_i and E_j are Pin^c, and E_k is orientable, then E_k has a natural Spin^c structure.

Example 8. The real projective spaces $\mathbb{R}P^{4k}$, $\mathbb{R}P^{4k+2}$, and $\mathbb{R}P^{4k+3}$ are all Pin^c. However, the latter is also Spin, unlike the former two, since real projective spaces of even dimension are non-orientable.

Example 9. $\mathbf{Wu}^c$ structure in M-theory. Consider M-theory on an 11-dimensional Spin manifold Y^{11} with a C-field, as described in [6]. From the inclusion $H^4(Y^{11};\mathbb{Z}_2) \hookrightarrow H^4(Y^{11},\mathbb{R}/\mathbb{Z}) \hookrightarrow \check{H}^4(Y^{11})$, the cohomology class $w_4(Y^{11}) \in H^4(Y^{11};\mathbb{Z}_2)$ defines a differential cohomology class $\check{w}_4$. The characteristic class of this flat character is the integral class $W_5(Y^{11}) = \beta w_4(Y^{11})$. This class is interpreted [6] as the background magnetic charge induced by the topology of Y^{11} and should vanish to be able to formulate the C-field. On Spin manifolds, $W_5(Y^{11}) = 0$ since the class λ is the integral lift of $w_4(Y^{11})$. Now, on a Spin manifold we have the fourth Wu class $v_4 = w_4$, so that we have as condition the vanishing of the integral lift of the Wu class v_4, i.e.

$$V_5(Y^{11}) = \beta v_4(Y^{11}) = 0 \, . \tag{5.1}$$

Therefore, the C-field in M-theory leads to a $\mathrm{Wu}(4)^c$ structure.

For manifolds of dimension $4k$, the characteristic elements for the intersection pairing in the middle dimension are the integer lifts λ of the Wu-class v_{2k}.

Example 10. $\mathbf{Wu}^c$ structure for the M5-brane. The study of the partition function of the M5-brane in [32] [11] lifted to eight dimensions requires a mod 2 middle cohomology class to lift to an integral class. If the M5-brane worldvolume if not Spin (this happens often) then instead of $\beta w_4 = 0$ we will have $\beta v_4 = 0$, that is $W_5 - \beta(w_2^2) = 0$. This defines a $\mathrm{Wu}(4)^c$ structure. We can also interpret this as a twisted $\mathrm{Membrane}^c$ structure.

Mod 2 reduction and Wu^c structure. Since the Steenrod square, the mod 2 reduction, and the Bockstein are related as $Sq^1 = \rho_2\beta$, we have $Sq^1 v_i = \rho_2(\beta v_i)$, so we have that a Wu^c structure implies that $Sq^1 v_i = 0$ in that degree. This latter condition appears naturally when considering Wu classes of Spin bundles (see Appendix in [11]): $Sq^1 v_{4k}(E) = 0$ for a Spin bundle E. Conversely, this condition, which is always satisfied in the Spin case, implies that the mod 2 reduction of the Wu^c class in degree $4k+1$ is zero. For example, let E be a real vector bundle over a space X and consider the degree four Wu class v_4. Applying Sq^1 gives $Sq^1 v_4 U = Sq^1 \chi(Sq^4)U$, which by property of χ gives $\chi(Sq^4 Sq^1)U$. Now using the Adem relation $Sq^2 Sq^3 = Sq^4 Sq^1$, this gives $\chi(Sq^2 Sq^3)U$. Applying the definition of χ leads to $\chi(Sq^3)Sq^2 U$. Since the bundle is assumed to be Spin we have $Sq^2 U = w_2 U = 0$, so that $Sq^1 v_4 = 0$.

We now consider the integral lifts of the Wu classes. We seek to characterize classes x such that $\rho_2(x) = v$, according to degrees. The discussion will proceed according to whether the degrees are of the form $4k$ or $4k+2$. We start with the first case and illustrate for degree 4 and degree 8, the degrees which seem most relevant for applications.

String^c structures and integral lifts of the Wu class v_4. A String^c structure is defined by obstruction [4]

$$Q_1 + c^2 = 0 \in H^4(M; \mathbb{Z}) \ , \tag{5.2}$$

where c is the first Chern class of the complex line bundle which defines the Spin^c structure. On the other hand, in the oriented case the Wu class v_4 is given in terms of the Stiefel-Whitney classes as $v_4 = w_4 + w_2^2$. Since w_4 is the mod 2 reduction of the first Spin class $Q_1 = \frac{1}{2}p_1$ and w_2 is the mod 2 reduction of the first Chern class c, we have that $w_4 + w_2^2$ is the mod 2 reduction of the $Q_1 + c^2$. We have used the formula $w_2(M) \cup c = c \cup c$ mod 2 in the case when our manifold is oriented, with $c \in H^2(M; \mathbb{Z})$. Therefore, we have found an integral lift of the Wu class in this case

Proposition 8 *An integral lift of a Wu(4)-structure on oriented 4-manifolds is given by a String^c structure.*

Fivebrane$^{K(\mathbb{Z},4)}$ structures and integral lifts of the Wu class v_8. Now we consider a Fivebrane$^{K(\mathbb{Z},4)}$ structure, defined by the condition [21]

$$Q_2 + e^2 = 0 \in H^8(M; \mathbb{Z}) \ , \tag{5.3}$$

where e is the degree four class of a $K(\mathbb{Z}, 3)$ bundle. In the Spin case we have $v_8 = w_8 + w_4^2$, since $w_4 \cup e = e \cup e$ mod 2. Therefore, similarly to the degree 4 case above, we have

Proposition 9 *An integral lift of a Wu(8)-structure on Spin 8-manifolds is given by a Fivebrane$^{K(\mathbb{Z},4)}$ structure.*

We now consider the Wu classes of degree $4k+2$.

Squares of odd-dimensional Steifel-Whitney classes and torsion Pontrjagin classes. The Wu classes in degrees $2j$ contain squares of classes of degree j, i.e. $(w_j)^2$. For low degree cases, this can be seen explicitly from (2.8). We already understand the structures implied by the

indecomposable Stiefel-Whitney classes (and hence of corresponding Wu classes), at least in lower degrees. The most notable decomposable terms not involving w_1 are the squares mentioned above. Hence, we would like to gain a better understanding of such terms. For j odd, we do this using torsion Pontrjagin classes [30]. These classes are $\mathcal{P}_{4k+2}$ with an indexing such that $\mathcal{P}_{4i} = p_i$ coincide with the usual ith Pontrjagin classes. For the degrees $4k+2$ (hence corresponding to half-integer indexing had we kept the usual notation) are 2-torsion: $2\mathcal{P}_{4k+2} = 0 \in H^{4k+2}\mathbb{Z}$. One way to define these classes for a vector bundle E is via the action of the Bockstein and the Steenrod square on the Stiefel-Whitney classes in degree $2k+1$, that is $\mathcal{P}_{4k+2}(E) = \beta Sq^{2k} w_{2k+1}(E)$. The mod 2 reduction $\rho_2 : H^{4k+2}(X;\mathbb{Z}) \to H^{4k+2}(X;\mathbb{Z}_2)$ of these classes gives precisely the desired squares of Stiefel-Whitney classes $\rho_2 \mathcal{P}_{4k+2}(E) = w_{2k+1}(E)^2$. Therefore, in the situation where the the Wu class is given by the squares, we can find an integral lift.

Proposition 10 *In the situation described above, the integral lifts of the Wu classes are the torsion Pontrjagin classes.*

Example 11. We illustrate the proposition in low degree examples.
1. *Degree two*: $v_2 = w_2 + w_1^2$. If $w_2 = 0$, that is we have a Pin^+ structure, then a lift of the Wu(2) structure is given by the torsion Pontrjagin class $\mathcal{P}_2$. We could also be in a situation where $w_2 = \rho_2(c_1)$, where c_1 is the first Chern class.

2. *Degree six*: In the oriented case we have $v_6 = w_2 w_4 + w_3^2$. If $w_4 = 0$, i.e. if we have a Membrane structure [21] then v_6 reduces to the square term w_3^2. Then we see that the integral lift of the Wu class in this case if the torsion Pontrjagin class $\mathcal{P}_6$. Note that we cannot instead set $w_2 = 0$ as then w_3 would also be zero.

3. *Degree ten*: Here v_{10} will involve v_2, due to the formula (3.4). Hence we cannot possibly isolate a square term. However, if such a term is present then it will be given by the torsion Pontrjagin class $\mathcal{P}_{10}$. Note that higher degree Stiefel-Whitney classes can be disposed of, for example when considering an oriented 11-manifold Y^{11} for which $w_{11}(Y^{11}) = w_{10}(Y^{11}) = w_9(Y^{11}) = 0$ (by using [14]). This gets rid of any possible terms involving w_{10} and w_9 in v_{10}.

6 Twisted Wu^c structures

In this section we will take the Wu^c structures we defined in the previous section and give them a twist. That is, we will consider a slight relaxation of the Wu^c condition. A twisted Wu$(i)^c$ structure will be defined by the condition $\beta v_i + \alpha_{i+1} = 0$, where α_{i+1} is a degree $i+1$ integral class. More precisely, we have

Definition 11 *Let (X, α_{i+1}) be a compact topological space with a degree $i+1$ integral cocycle $\alpha_{i+1} : X \to K(\mathbb{Z}, i+1)$. A twisted Wu^c structure over X is a quadruple (M, ν, ι, η), where*
(1) M is a smooth compact oriented manifold together with a fixed classifying map of its stable normal bundle $\nu : M \to BO$;
(2) $\iota : M \to X$ is a continuous map;
(3) η is an α-twisted structure on M defined by $V_{i+1}(M) + \iota^* \alpha_{i+1} = 0$, that is, a homotopy com-

mutative diagram

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & BO[v_j] \arrow[d, "V_{i+1}"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "\alpha_{i+1}"'] & K(\mathbb{Z}, i+1) \end{tikzcd} , \tag{6.1}$$

where η is a homotopy between $w_i \circ \nu$ and $\alpha \circ \iota$.

Remarks. 1. The idea of the twisted $\mathrm{Wu}(i)^c$ structure is that the Wu class might not be the mod 2 reduction of an integral class exactly, but only so up to an (auxiliary) integral class.

2. If the composite Stiefel-Whitney classes all vanish, then a twisted $\mathrm{Wu}(i)^c$ structure reduces to a twisted Pin^c structure, twisted $\mathrm{Membrane}^c$ structure, and twisted $\mathrm{String}^{(K(\mathbb{Z},3)}$ structure for $i = 2, 4$, and 6, respectively. The latter two are defined in [21], and so we now give a definition of the first.

Definition 12 *A twisted Pin^c structure is a structure given in definition 11 for $i = 2$.*

When the twist is zero, this reduces to the usual Pin^c structure.

We now consider a sample property of the twisted Wu^c structures. The obstructions to twisted Wu^c structures are additive for bundles

$$\begin{aligned} (\beta v + \alpha)(E \oplus F) &= \beta v(E \oplus F) + \alpha(E \oplus F) \\ &= \beta v(E) + \beta v(F) + \alpha(E) + \alpha(F) \ , \end{aligned}$$

provided that α is an additive class, as then we have

$$\beta v(E \oplus F) = \beta v(E) + \beta v(F) \ . \tag{6.2}$$

Proposition 13 *Wu^c structures are additive.*

We will find that for our applications we need a relative version of twisted Wu^c structures. Relative versions of classical Wu classes are defined in [12], on which we build our generalization.

Relative Wu classes. Consider a manifold Z with boundary Y and corresponding tangent bundles $\pi : TZ \to Z$ and $p : TY \to Y$. The relative second Wu class $v_2(Z, Y)$ can be defined via the diagram

$$\begin{tikzcd} H^1(TY; \mathbb{Z}_2) \arrow[r, "\delta_T"] & H^2(TZ, TY; \mathbb{Z}_2) \\ H^1(Y; \mathbb{Z}_2) \arrow[u, "\pi^*"] \arrow[r, "\delta"] & H^2(Z, Y; \mathbb{Z}_2) \arrow[u, "\pi^*"', "\cong"] \end{tikzcd} \tag{6.3}$$

i.e. is the element $(\pi^*)^{-1} \circ \delta_T(\sigma)$, where σ is an element of the set of Pin^- structures on TY. If $j : (Z, \emptyset) \to (Z, Y)$ is the inclusion then $j^* v_2(Z, Y; \sigma) = v_2(Z)$ is the usual class for each σ in the set of Pin^- structures $\mathrm{Wu}_2(Y)$. The following proposition is a straightforward extension of a result in [2] in the Spin case.

Proposition 14 *The function $v_2(Z,Y) : \mathrm{Wu}_2(Y) \to H^2(Z,Y;\mathbb{Z}_2)$ satisfies the following:*
(i) image$(v_2(Z,Y)) = (j^*)^{-1}(v_2(Z))$;
(ii) if Z is 4-dimensional, a class $v \in H^2(Z,Y;\mathbb{Z}_2)$ lies in the image of $v_2(Z,Y)$ if and only if $\langle v, j_*(\xi)\rangle \equiv \xi \cdot \xi$ (mod 2) for each $\xi \in H_2(Z)$;
(iii) if $x \in H^1(Y;\mathbb{Z}_2)$ and $\sigma \in \mathrm{Wu}_2(Y)$, then

$$v_2(Z,Y;x\cdot u) = v_2(Z,Y;\sigma) + \delta x.$$

Thus $v_2(Z,Y)$ is injective;
(iv) if Z is 4-dimensional, then $v_2(Z,Y;\sigma) = 0$ if and only if the intersection pairing on the middle homology of Z is even and $\sigma = \sigma_Z$.

The proof of (ii) follows from (i) and an application of the Wu formula. The latter also establishes the rest of the statements.

Note that by Lefschetz duality $D : H^i(Z,Y;\mathbb{Z}_2) \to H_{n-i}(Z;\mathbb{Z}_2)$ one gets an absolute homology class which is dual to the relative cohomology class; hence for some purposes it might be more convenient to work with homology. We use this in the propsitions below; see [22] for more extensive applications.

The above discussion on relative Pin^- structures can be generalized to higher Wu structures. Our examples will keep us grounded and consider relatively low degree cases instead of aiming for utmost generality. We will need the following to consider the M5-brane case.

Relative Wu class on eight-manifolds. Consider an 8-manifold Z^8 with boundary Y^7 and tangent bundles $\pi : TZ^8 \to Z^8$ and $p : TY^7 \to Y^7$. The relative fourth Wu class $v_4(Z^8, Y^7)$ can be defined via the diagram (6.3), with the obvious changes in the degrees of the cohomology groups, This is the element $(\pi^*)^{-1} \circ \delta_T(\sigma)$, where σ is an element of $\mathrm{Wu}_4(Y^7)$, the set of Wu_4 structures on TY^7. Similarly, we have

Proposition 15 *The function $v_4(Z^8,Y^7) : \mathrm{Wu}_4(Y^7) \to H^4(Z^4,Y^7;\mathbb{Z}_2)$ satisfies the following:*
(i) image$(v_4(Z^8,Y^7)) = (j^*)^{-1}(v_4(Z^8))$;
(ii) a class $v \in H^4(Z^8,Y^7;\mathbb{Z}_2)$ lies in the image of $v_4(Z^8,Y^7)$ if and only if $\langle v, j_*(\xi)\rangle \equiv \xi \cdot \xi$ (mod 2) for each $\xi \in H_4(Z^8)$;
(iii) $v_4(Z^8,Y^7;\sigma) = 0$ if and only if the intersection pairing on the middle homology of Z^8 is even and $\sigma = \sigma_Z$.

Example 12. Type IIB string theory. Similar results holds for the 6th Wu class on 12-manifolds with boundary, relevant to type IIB string theory.

We now need a notion of relative Wu^c-strucutre. A straightforward application of the general formulation in [12] to our Wu^c classes (cf. Definition 6) gives the following defintion

Definition 16 *For a pair of spaces (Z,Y), we define a relative Wu^c structure by applying the Bockstein on the relative Wu classes, that is $V_{i+1}(Z,Y) = \beta v_i(Z,Y)$.*

Next, we add the twist to our relative Wu^c structures.

Definition 17 *A relative twisted Wu^c) structure on a pair (Z, Y) is described by the diagram*

$$\begin{tikzcd} (Z,Y) \arrow[r, "f"] \arrow[dr, "\alpha_{i+1}"'] & BO[v_j] \arrow[d, "V_{i+1}"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & K(\mathbb{Z}, i+1) \end{tikzcd} \quad , \tag{6.4}$$

where η is a homotopy between the map representing the relative integral class $V_{i+1}(Z, Y)$ and the relative cocycle α_{i+1}.

Remarks. 1. As in previous defintions, such a twisted notion of Wu structure can be defined on a brane M with the (by now) obvious changes.

2. The condition for existence of a relative twisted Wu^c structure of degree i is $V_{i+1}(Z, Y)+\alpha_{i+1} = 0 \in H^{i+1}(Z, Y; \mathbb{Z})$.

With the above defintions and results, we now present our main application of relative twisted Wu^c structures.

Example 13. The M5-brane. Consider the worldvolume of the M5-brane as a 6-manifold M^6. In the usual Chern-Simons construction, this is the base of a circle bundle Y^7, which is a boundary of an 8-manifold Z^8. The fields are considered to be cohomology classes of degree three (i.e. middle-degree) on M^6 which lift to degree four cohomology classes on Z^8. In addition to the usual intersection pairing on M^6, we have a torsion pairing on Y^7. Let $T^4(Y^7)$ denote the torsion subgroup of the integral cohomology group $H^4(Y^7; \mathbb{Z})$, and similarly let $T^4(Z^8)$ be the torsion subgroup of $H^4(Z^8; \mathbb{Z})$. The torsion pairing is given by the symmetric bilinear nonsingular pairing

$$L : T^4(Y^7) \times T^4(Y^7) \to \mathbb{Q}/\mathbb{Z} \ . \tag{6.5}$$

Let $i : T^4(Z^8) \to T^4(Y^7)$ be the inclusion. This is the adjoint with respect to the pairing L of the map $\delta : T^4(Y^7) \to T^5(Z^8, Y^7)$, that is $L(i(x), y) = \langle x, \delta y\rangle$, where $x \in T^4(Z^8)$ and $y \in T^4(Y^7)$.

A lifting $\hat{v} \in H^4(Z^8, Y^7; \mathbb{Z}_2)$ of the Wu class $v_4 \in H^4(Z^8; \mathbb{Z}_2)$ is said to be compatible with the quadratic form $\psi : T^4(Y^7) \to \mathbb{Q}/\mathbb{Z}$ if, for all $x \in T^4(Y^7)$ such that $x = i^*(y)$, we have

$$\psi(x) = \frac{1}{2}\langle y \cdot \hat{v}_4, [Z^8, Y^7]\rangle - \frac{1}{2}\langle y \cdot (j^*)^{-1}y, [Z^8, Y^7]\rangle \in \mathbb{Q}/\mathbb{Z} \ . \tag{6.6}$$

Let $j^* : H^4(Z^8, Y^7; \mathbb{Z}) \to H^4(Z^8; \mathbb{Z})$ be the map induced from the map that forgets the boundary. Given a quadratic function on $T^4(Y^7)$, there is a corresponding Wu class $\hat{v}_4 \in H^4(Z^8, Y^7; \mathbb{Z}_2)$. Now let $b_4 \in T^4(Y^7)$ be class such that $\psi i(y) = L(b_4, i(y))$ for all $y \in T^4(Z^8)$. Then we have

$$\beta\hat{v}_4 = \delta^* b_4 \ , \tag{6.7}$$

where $\beta : H^4(Z^8, Y^7; \mathbb{Z}_2) \to H^5(Z^8, Y^7; \mathbb{Z})$ is the integral Bockstein and $\delta^* : H^4(Y^7; \mathbb{Z}) \to H^5(Z^8, Y^7; \mathbb{Z})$ is the coboundary map of the pair (Z^8, Y^7).

We now interpret expression (6.7) as defining a twisted (relative) Wu(4)^c structure.

There exists an integral class $v_4' \in H^4(Z^8;\mathbb{Z})$ such that the mod 2 reduction is the absolute (i.e. non-relative) Wu class $\rho_2(v_4') = v_4 \in H^4(Z^8;\mathbb{Z}_2)$ and $i^*(v_4') = 2b_4 \in H^4(Y^7;\mathbb{Z})$.

Consider the Wu relation $Sq(v) = w$, where Sq is the total Steenrod square operation, v is the total Wu class and w is total Stiefel-Whitney class. The degree 4 component gives $v_4 = w_4+$ lower degree classes. If we require these lower degrees to vanish, then the Wu class and the Stiefel-Whitney class coincide in degree 4. Therefore, the above condition can be written as

$$\beta w_4 = \delta^* b \ , \tag{6.8}$$

which is of the form $W_5 - H_5 = 0$ (albeit in relative cohomology).

Example 14. Type IIB string theory. The structure of type IIB string theory in ten dimensions, on M^{10}, is in some ways very similar to that of the M5-brane. In particular, here we use for fields cohomology classes of degree five in ten dimensions which lift to cohomology classes of degree six in twelve dimensions, on Z^{12} with $\partial Z^{12} = Y^{11}$. In this case, the above construction yields the condition in degree seven

$$\beta \hat{v}_6 = \delta^* b_6 \ , \tag{6.9}$$

where $b_6 \in T^6(Y^11)$ be class such that $\psi i(y) = L(b_6, i(y))$ for all $y \in T^6(Z^{12})$.

We now interpret expression (6.9) as defining a twisted (relative) $\mathrm{Wu}(6)^c$ structure.

There exists an integral class $v_6' \in H^6(Z^{12};\mathbb{Z})$ such that the mod 2 reduction is the absolute (i.e. non-relative) Wu class $\rho_2(v_6') = v_6 \in H^6(Z^{12};\mathbb{Z}_2)$ and $i^*(v_6') = 2b_6 \in H^6(Y^{11};\mathbb{Z})$.

We record the results of the above two examples in the following

Proposition 18 *The lift of the Wu class v_i in the sense of [3] gives rise to a twisted Membrane^c structure and a twisted String$^{K(\mathbb{Z},3)}$ structure on the 8-manifold and the 12-manifold for $i = 4$ and $i = 8$, respectively.*

Example 14 is also discussed in [22] in relation to global anomalies in type IIB string theory.

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